Quantum optical thermodynamic machines: Lasing as relaxation

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(Received 11 May 2009; revised manuscript received 26 October 2009; published 23 December 2009)

Motivated by the growing interest in the nanophysics and the field of quantum thermodynamics we study an open quantum system consisting of two spatially separated two-level atoms (spins) coupled to a quantum oscillator (resonator field mode). There is no external driving. The spins of different energy splittings are each linked to a heat bath with different temperature. We find that the temperature gradient imposed on the system together with the oscillator operating as a kind of work reservoir makes this system act as a thermodynamic machine, in particular, as a heat engine (laser). We analyze the properties of the resulting resonator field and of the engine functionality. For the latter problem we use recently developed definitions of heat flux and power as well as a test, in which the resulting field is used as an input for a heat pump.

DOI: 10.1103/PhysRevE.80.061129 PACS number(s): 05.70.Ln, 42.60.Jf, 42.50.−p

I. INTRODUCTION

Quantum thermodynamics [1] is a growing area of research, merging two important fields, quantum theory, and thermodynamics, which represent an essential part of our understanding of nature around us. This merging is not superficial but imposed by the continuing advances of technology and the attempts to make devices smaller and smaller.

The quantum heat engine [1] is one of those tools by which we can test this merging. Thermodynamic machines [2] (heat engine and heat pump) are prototype models to apply and test thermodynamic fundamental laws. Typically they consist of a working medium (a gas enclosed in a cylinder), and a work reservoir (acting on a movable piston or any other mechanical degree of freedom). The working medium can alternatively be kept isolated or brought in thermal contact with one of two heat baths at different temperatures. The basic engine operation has four strokes, but continuous operation is also possible (see below).

In order to quantize the heat engine, we can either quantize the working medium, or quantize the work reservoir or quantize both. Typically, the heat baths are assumed to be macroscopic and to have a thermal equilibrium Gibbs distribution. In the seminal work of Scovil and Schulz-DuBois [3], a three-level maser has been analyzed as a heat engine. Based on the Boltzman distribution of the atomic population they showed that the efficiency is less or equal to the Carnot efficiency [2]. Alicki [4] studied a quantum open system weakly coupled to thermal reservoir at different temperatures as a model of heat engine; by partitioning the energy of the system into heat and work he obtained the Carnot inequality for the efficiency of the heat engine. The possibility of such an open quantum system to perform mechanical work was originally introduced by Pusz and Wornowicz [5] by modeling the change in the external conditions (e.g., switching on some external fields or moving the walls confining the space accessible to the system) with a family of self-adjoint operators. By changing such external conditions we can transmit energy to or from the system (apply or extract work).

Recently, several quantum heat engines have been proposed (see [6–26] and reference therein), where many fundamental problems were discussed. However, a generally accepted definition of work and heat for autonomous systems (no external driving) has not yet been reached. In this paper, we discuss this problem by applying two recently suggested definitions [6,7] to a our scenario. Preliminary results have been published as a conference paper [27]. The debate on work versus heat should be seen here in the context of the appropriate characterization of the output laser field: how to measure its usefulness?

II. MODEL AND ITS TIME EVOLUTION

The considered engine, Fig. 1, is an open quantum system consisting of two spin-1/2 (atoms) A and B with different

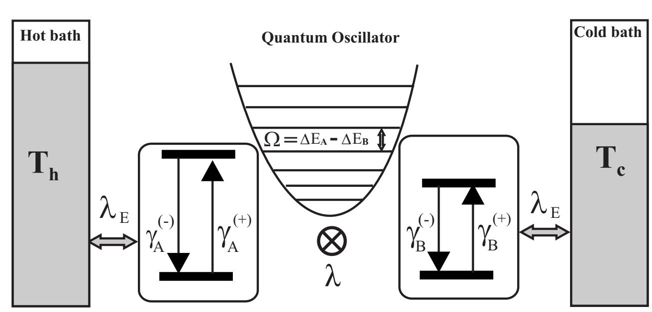

FIG. 1. Schematic diagram of the model. The spatially separated two-level atoms A and B (working medium) are in contact with heat bath T_h and T_c, respectively, and coupled via a single cavity photon mode Ω, the work reservoir.

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splittings $\Delta E_A \geq \Delta E_B$ (the quantized working medium), placed in a closed cavity and locally in contact with two separate heat reservoirs at different temperature: spin A is coupled to a hot temperature bath T_h and spin B is coupled to a cold temperature bath T_c. These reservoirs induce relaxation processes with a nearest neighbor selection rule (local-site coupling), which would bring each spin individually to a canonical equilibrium state of temperature $T_{h(c)}$.

The two spins interact through a single cavity mode (a photon assisted interaction), which represents the "quantum work reservoir," such that the difference between the energy splitting of the two atoms is in resonance with the quantized cavity mode. The dynamics of the matter-field system is described by the following Hamiltonian:

$$\hat{H}_S = \hat{H}_A + \hat{H}_B + \hat{H}_f + \hat{H}_{int}. \tag{1}$$

$$\hat{H}_S = \Omega \hat{a}^\dagger \hat{a} + \frac{\Delta E_A}{2}\hat{\sigma}_A^z + \frac{\Delta E_B}{2}\hat{\sigma}_B^z + \lambda(\hat{\sigma}_A^- \hat{\sigma}_B^+ \hat{a}^\dagger + \hat{\sigma}_A^+ \hat{\sigma}_B^- \hat{a}). \tag{2}$$

The units have been chosen such that $\hbar = 1$. The terms $\hat{H}_{A(B)}$ and $\hat{H}_f$ represent the local Hamiltonians of the atom A(B) and the quantized cavity mode of frequency Ω, respectively, while $\hat{H}_{int}$ is the interaction Hamiltonian. Here, we consider the resonance interaction, i.e., $\Omega = \Delta E_A - \Delta E_B$. The operators, $\hat{\sigma}_i^+ = |e\rangle\langle g|$, $\hat{\sigma}_i^- = |g\rangle\langle e|$, and $\hat{\sigma}_i^z = (|e\rangle\langle e| - |g\rangle\langle g|)$, $i = A, B$, are the usual raising, lowering, and the inversion operators for atom A(B), respectively, where $|e(g)\rangle$ are the energy eigenstates of the local Hamiltonians $\hat{H}_i$. They act on their respective Hilbert spaces and satisfy $[\hat{\sigma}_{A(B)}^+, \hat{\sigma}_{A(B)}^-] = \sigma_{A(B)}^z$, $[\hat{\sigma}_{A(B)}^z, \hat{\sigma}_{A(B)}^\pm] = \pm 2\sigma_{A(B)}^\pm$, $[\hat{\sigma}_{A(B)}^j, \hat{\sigma}_{B(A)}^j] = 0$. The operators $\hat{a}^\dagger$ and $\hat{a}$ are the Bose creation and annihilation operators for the quantized field mode, satisfying the commutation relations $[\hat{a}, \hat{a}^+] = 1$, and λ is the interaction coupling constant.

Hamiltonian (1) conserves the total number of excitations of the atoms plus the field. This suggests a decomposition for the system Hilbert space in the product basis states $\mathcal{H} = \Sigma_{n=0}^{\infty} \oplus \mathcal{H}_n$ such that $\mathcal{H}_n|_{n=0}^{\infty} = \{|e,e;n\rangle, |e,g;n\rangle, |g,e;n+1\rangle, |g,g;n\rangle\}, n = 0, 1, 2, \ldots$, where $|e\rangle(|g\rangle)$ denotes the upper (lower) state of the atoms, and $|n\rangle$ is the Fock state of the field mode with n photons.

Our open system is described by the Liouville–von Neumann equation [28],

$$\frac{\partial \hat{\rho}_s(t)}{\partial t} = -i[\hat{H}_S, \hat{\rho}_s(t)] + \hat{\mathcal{L}}_A[\hat{\rho}_s(t)] + \hat{\mathcal{L}}_B[\hat{\rho}_s(t)]$$
$$= \hat{\mathcal{L}}_{coh}[\hat{\rho}_s(t)] + \hat{\mathcal{L}}_D[\hat{\rho}_s(t)], \tag{3}$$

where $\hat{\rho}_s(t)$ is the density operator and $\hat{\mathcal{L}}_{coh}(\hat{\rho})$ is the Hamiltonian superoperator, which governs the coherent dynamics of the system. The second and third Liouville superoperators model the influence of the two heat baths (e.g., thermal photons). In the system-environment weak coupling regime and under Born-Markov approximation these dissipators can be written in the well-known Lindbald form,

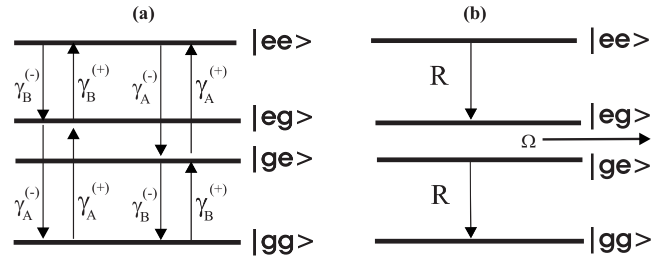

FIG. 2. (a) Population flows due to the various terms in the master equation (3). (b) Laser action taking place between the two excited energy levels $|e,g\rangle$ and $|g,e\rangle$ separated by a frequency Ω. Level $|e,g\rangle$ is effectively excited at rate R, while level $|g,e\rangle$ effectively decays with the same rate.

$$\hat{\mathcal{L}}_{A(B)}(\hat{\rho}_s) = \sum_{\alpha=\pm} \gamma_{A(B)}^{\alpha}\left(\hat{L}_{A(B)}^{\alpha}\hat{\rho}\hat{L}_{A(B)}^{\alpha^\dagger} - \frac{1}{2}\{\hat{L}_{A(B)}^{\alpha^\dagger}\hat{L}_{A(B)}^{\alpha}, \hat{\rho}_s\}\right). \tag{4}$$

The braces {,} represent the anticommutator. The operators $\hat{L}_{A(B)}^{\alpha}$ belong to the Hilbert space of the system, and for a spin A(B) coupled to a reservoir, $\hat{L}_{A(B)}^{\alpha} = \hat{\sigma}_{A(B)}^{\alpha}$. The positive constants γ^α are the rates of the two damping channels, which determine the equilibrium state of the relaxing system "atom A(B)" and, hence, its temperature. They are related to the Weiskopf-Wigner [29,30] decay constants Γ_A and Γ_B associated with the hot and cold reservoirs by the relations $\gamma_{A(B)}^- = \Gamma_{A(B)}(\bar{n}_{h(c)} + 1)$ and $\gamma_{A(B)}^+ = \Gamma_{A(B)}\bar{n}_{h(c)}$, where $\bar{n}_{h(c)} = (\exp[\frac{\Delta E_{A(B)}}{k_B T_{h(c)}}] - 1)^{-1}$ are the number of thermal photons in the $h(c)$ baths and k_B is the Boltzmann constant. We will consider that each atom is coupled to the relevant bath with the same strength, so it is convenient to parametrize the atom-bath coupling strength as

$$\lambda_E = \gamma_A^- + \gamma_A^+ = \gamma_B^- + \gamma_B^+. \tag{5}$$

Consequently, we have the following explicit expressions for the rates in terms of the atom-bath coupling strength, atom's energy splitting, and the bath as follows [see Fig. 2(a)]:

$$\gamma_{A(B)}^+ = \lambda_E \frac{1}{1 + e^{\beta_{h(c)}\Delta E_{A(B)}}},$$

$$\gamma_{A(B)}^- = \lambda_E \frac{1}{1 + e^{-\beta_{h(c)}\Delta E_{A(B)}}}, \tag{6}$$

where $1/\beta_{h(c)} = T_{h(c)}$ and we set $k_B = 1$. Temperature is thus measured in energy units. To solve Eq. (3) we convert the operator equation into a set of coupled ordinary differential equations using the product basis states (see Appendixes A and B). These are solved by the fourth-order Runge-Kutta method. Also we use the quantum optics toolbox for MATLAB [31].

III. LASER ACTION

The laser principle is concerned with the creation of coherent light through an ensemble of atoms driven far away from thermal equilibrium via a continuous flux of energy through the system [32]. In the standard laser model N two-level atoms interact with one mode of the resonator, and to obtain a laser action, a pump process is included to get ground state atoms to the upper lasing level. Typically, the atoms being regularly injected into the cavity (the cavity mode is resonant with the atomic dipole-allowed transition) and the pumping is either coherent, e.g., the atoms interact with an external radiation before entering the cavity, or incoherent e.g., the field mode interacts with a reservoir of inverted atoms (negative effective temperature [33], T_{eff}.

In our model, Fig. 1, the pumping process is achieved by connecting the system to two reservoirs at different (positive) temperatures, the states $|e,g\rangle$ and $|g,e\rangle$ are the upper and lower lasing levels with occupation probabilities P_{eg},P_{ge}, respectively. Laser action occurs, when $P_{eg}>P_{ge}$ i.e., $R=(\gamma_B^{(-)}-\gamma_B^{(+)})-(\gamma_A^{(-)}-\gamma_A^{(+)})>0$ [see Fig. 2(a)]. One can show that this condition is equivalent to

$$1<\frac{T_h}{T_c}\geq\frac{\Delta E_A}{\Delta E_B}. \tag{7}$$

This scenario may alternatively [34] be viewed as if there was a beam of two-level atoms (atomic reservoir) of "negative temperature" [33] T_{eff}, i.e., a beam, in which there are more atoms in the excited state than in the ground state [34], according to a Boltzmann distribution, Fig. 2(b), with negative T_{eff},

$$\frac{P_{eg}}{P_{ge}}=\exp\left[\frac{\Delta E_A-\Delta E_B}{k_BT_{eff}}\right]=\exp\left[\frac{\Omega}{k_BT_{eff}}\right], \tag{8}$$

consequently, the field is amplified. On the other hand, when

$$1<\frac{T_h}{T_c}<\frac{\Delta E_A}{\Delta E_B}, \tag{9}$$

there are more atoms in the ground state than in the excited state according to a Boltzmann distribution with positive temperature T_{eff} given by

$$\frac{P_{eg}}{P_{ge}}=\exp\left[-\frac{\Delta E_A-\Delta E_B}{k_BT_{eff}}\right]=\exp\left[-\frac{\Omega}{k_BT_{eff}}\right]. \tag{10}$$

In this case the field is damped by these atoms.

Following [29,35,36], in Appendix B we derive the equation of motion for the laser field density matrix due to the interaction with the lasing medium. For the diagonal elements $\rho^{n;n}=P(n)$, which represent the probability for n photons in the field mode, we have

$$\dot{P}(n)=\bar{\lambda}\Pi_1nP(n-1)-\bar{\lambda}\Pi_2nP(n)-\bar{\lambda}\Pi_1(n+1)P(n)$$
$$+\bar{\lambda}\Pi_2(n+1)P(n+1). \tag{11}$$

Comparing this equation with the standard laser equation [29,37], we interpret the parameter $\bar{\lambda}=\frac{\lambda^2}{\lambda_E}$ as the effective pumping rate, the parameter $\bar{\lambda}\Pi_1$ as the gain coefficient, $\bar{\lambda}\Pi_2$ as the effective damping rate, and $\bar{\lambda}\Pi_1=\bar{\lambda}\Pi_2$ as the threshold condition for the laser. The term $\bar{\lambda}\Pi_1(n+1)P(n)$ represents the flow of probability from the $|n\rangle$ state to the $|n+1\rangle$ state due to the emission of photons by lasing atoms initially in the upper states: $\bar{\lambda}\Pi_1n$ is the rate of stimulated emission and $\bar{\lambda}\Pi_1$ is the spontaneous emission rate. Similarly, the term $\bar{\lambda}\Pi_2nP(n)$ represents the flow of probability from the $|n\rangle$ state to the $|n-1\rangle$, corresponding explanations exist for the other terms.

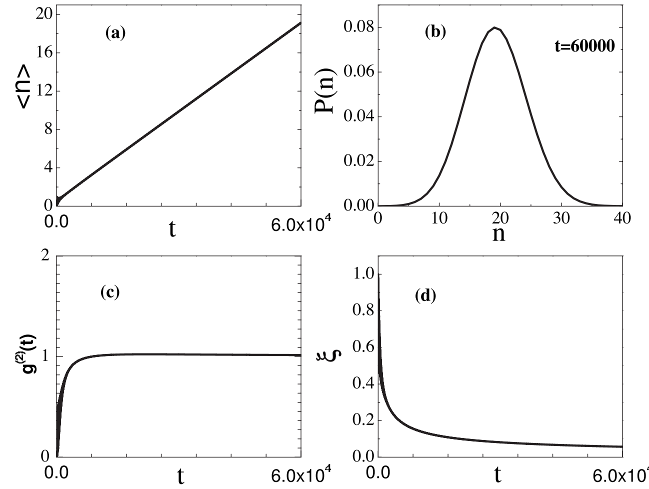

FIG. 3. Heat engine: (a) Average photon number, $\langle n\rangle$, (b) Momentary photon number distribution, $P(n)$, (c) Second-order correlation function, $g^2(t)$, and (d) Purity of the field mode, ξ. The parameters are $\lambda_E=0.001$, $\lambda=0.1$, $T_h=10$, $T_c=1$, $\Delta E_A=2$, and $\Delta E_B=1.8$. Time "t" is in inverse energy units $\times10^4$

IV. CHARACTERIZATION OF THE CAVITY FIELD UNDER LASING CONDITION

In order to statistically characterize the field mode inside the cavity, we turn to an examination of the time dependence of the field average photon number $\langle n\rangle$, the photon number distribution $P(n)$, the second-order correlation function $g^{(2)}(t)=\frac{\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle}{\langle\hat{a}^\dagger\hat{a}\rangle^2}$, the quasiprobability distribution Q-function for the field $Q(\alpha)=\langle\alpha|\hat{\rho}_f|\alpha\rangle$ and the field purity $\zeta=\mathrm{Tr}\{\hat{\rho}_f^2\}$, where $\hat{\rho}_f=\mathrm{Tr}_{AB}\{\hat{\rho}_s\}$ is the field reduced density operator.

In Figs. 3 and 4, we have taken the following parameters: $\lambda_E=0.001$, $\lambda=0.1$, $T_h=10$, $T_c=1$, $\Delta E_A=2$, $\Delta E_B=1.8$, and initial vacuum state for the field. The average photon number Fig. 3(a) inside the cavity increases linearly, due to the fact that there is an amplification without damping mechanism.

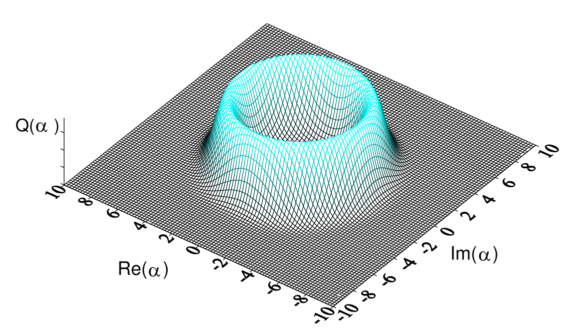

FIG. 4. (Color online) The Q function for the output field, typical of a laser far above threshold. Same parameters as in Fig. 3.

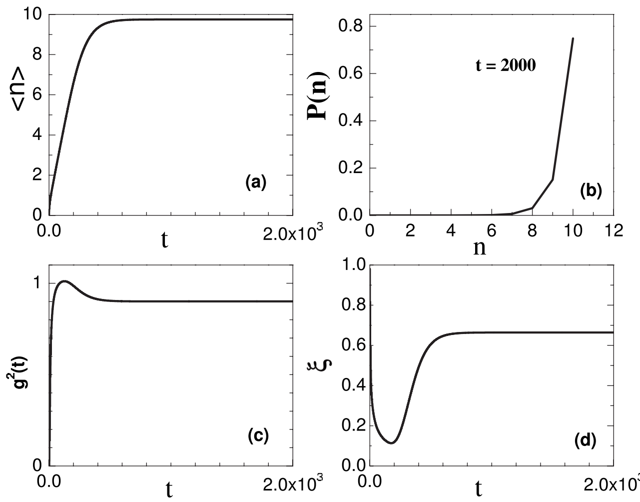

FIG. 5. Heat engine as of Fig. 3, but with λ_E=0.1, λ=1, and upper energy cutoff for the photon field: The Fock-space has a finite dimension N=10. Time "t" is in inverse energy units $\times 10^3$.

The second-order coherence function Fig. 3(c) indicates that the field is slightly super-Poissonian. Figure 3(b) shows that the photon statistics of the output is given by almost a Poisson distribution, which is a characteristic of a "coherent state." Thus, the output field has the same properties as the far above threshold laser, while for below threshold, the output would be essentially that of a black-body cavity (for a single mode).

It is important here to note, however, that the output in our model has no phase "coherence," in fact the Q-function shown in Fig. 5 is as reported previously (see e.g., [38,39]). It is centered on zero amplitude and its phase symmetrically distributed, thus the state is phase diffused. The field purity is shown in Fig. 3(d). It decreases with time and asymptotically will reach its minimum value. This means that the created field state is a maximally mixed state, but this mixedness is of a single mode field in the Fock state representation and should be distinguished from the mixedness of a field constructed from a broad spectrum of modes, which would be mixed in real space.

It was conjectured in [40] that optical coherence, i.e., quantum-mechanical coherence between states separated by Bohr frequencies in the optical regime, does not exist in the optical experiments. Also it has been argued that continuous-variable quantum teleportation at optical frequencies has not been achieved in [41], because the source used (a laser) was not "truly coherent," Many other discussions [42–46] were published around this subject. We will return to this point in the following sections.

Finally, we would like to mention that the above result is independent of the initial state of the field, as we have used Fock state, coherent state and a thermal state as an initial state of the field. This shows that the lasing process may be viewed as a relaxation process [47] toward the laser output state. This relaxation is, however, "frustrated" in the sense that the final stationary attractor cannot be reached, the average photon number continues to increase, the momentary field state is not a canonical (Gibbsian) state but a phase-diffused Glauber state. The respective stationary canonical state would have to have a negative temperature, which could be stabilized in a system with an upper energy bound only (see Fig. 5).

V. THERMODYNAMIC ANALYSIS

A laser can be viewed as a continuous thermodynamic device, which transforms heat into electromagnetic radiation [3]. Heat may be supplied in a variety of ways, ranging from glow discharge in cooled environments to absorption of incoherent light in a cold medium, or even irreversible cooling of gases by isentropic expansion (see [47] and reference therein).

Thermodynamics is based on two fundamental laws. The first law (the law of conservation of energy) reads

$$dE = \bar{d}Q + \bar{d}W, \tag{12}$$

where $\bar{d}Q$ is the infinitesimal change in the heat and $\bar{d}W$ is the infinitesimal change in the work. A differential form of the first law can be derived by calculating the expectation value of the energy of the respective quantum system

$$\dot{E} = \frac{d\langle\hat{H}\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}\{\hat{\rho}\hat{H}\} = \mathrm{Tr}\{\dot{\hat{\rho}}\hat{H}\} + \mathrm{Tr}\{\hat{\rho}\dot{\hat{H}}\} = \dot{Q} + \mathcal{P}, \tag{13}$$

where $\dot{Q}$ is the heat current and $\mathcal{P}$ is the power. In the last equality, the second identification is associated with the fact that work done on or by a system can be performed only through a change in the generalized coordinates [18] of the system, which in turn gives rise to a change of the energy spectrum.

The second law of thermodynamics, which limits our engine efficiency due to the entropy constraint, can be written as

$$dS = d_{ex}S + d_{in}S \geq 0, \tag{14}$$

where $d_{ex}S$ is the entropy supplied to the system by its surroundings and $d_{in}S$ is the entropy produced inside. If P_i is the occupation probability of the quantum state i (in equilibrium), the von Neumann entropy of the system is $S = -\Sigma_i P_i \ln P_i$. Therefore, a change dP_i is accompanied by a change in the entropy $dS = -\Sigma_i \ln P_i dP_i$ and, hence, the first identification.

This suggests that for an autonomous system no work can be done or extracted. While this holds for the whole quantum system, more detailed investigations are required for the characterization of the subsystem behavior.

Two different approaches will be analyzed here. First, based on Alicki's definition of quantum heat and work [4], Boukobza and Tannor [6,22] introduced a generalized definition of heat flux and power (work flux), applicable to any bipartite system. On the other hand, by computing the local energy expectation values with respect to some local measurement basis (LEMBAS) [7], the authors showed that for any quantum system there are two fundamentally different contributions: changes in energy that do not alter the local von Neumann entropy and changes that do. They identify the former as work and the latter as heat.

VI. WORK AND HEAT ACCORDING TO REF. [6]

The rate of change of energy of the total system is given by

$$\dot{E}_S = \frac{d\langle \hat{H}_S\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}\{\hat{\rho}_s\hat{H}_S\}$$
$$= \mathrm{Tr}\{\dot{\hat{\rho}}_s\hat{H}_S\} = -i\,\mathrm{Tr}\{[\hat{H}_{int},\hat{\rho}_s]\hat{H}_S\} + \mathrm{Tr}\{\hat{\mathcal{L}}_D(\hat{\rho}_s)\hat{H}_S\}$$
$$= \mathrm{Tr}\{\hat{\mathcal{L}}_D(\hat{\rho}_s)\hat{H}_S\} = \dot{Q}_h + \dot{Q}_c, \tag{15}$$

where $\dot{Q}_h, \dot{Q}_c$ are the heat currents associated with the hot and cold reservoir. The physical interpretation of Eq. (15) is that there is no work done on (by) the total system. The rate of change in energy of atom A is given by

$$\dot{E}_A = \frac{d\langle \hat{H}_A\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}_A\{\hat{\rho}_A\hat{H}_A\} = \mathrm{Tr}_A\{\dot{\hat{\rho}}_A\hat{H}_A\}, \tag{16}$$

where $\hat{\rho}_A = \mathrm{Tr}_{Bf}\{\hat{\rho}_s\}$ is the reduced density matrix for the atom A. Up to this point according to the energy partitioning of Alicki [4] and to the work definition of Pusz and Wornowicz [5] there is also no work associated with the atom A.

Following [6] and using some trace properties [48], we can write Eq. (16) in the following alternative form:

$$\dot{E}_A = \mathrm{Tr}\{\dot{\hat{\rho}}_s\hat{\mathbf{H}}_A\}, \tag{17}$$

where $\hat{\mathbf{H}}_A = (\hat{H}_A \otimes \hat{1}_B \otimes \hat{1}_f)$. Using Eq. (3) in the interaction picture we get

$$\dot{E}_A = -i\,\mathrm{Tr}\{\hat{\rho}_s[\hat{\mathbf{H}}_A,\hat{H}_{int}]\} + \mathrm{Tr}\{\hat{\mathcal{L}}_A(\hat{\rho}_s)\hat{\mathbf{H}}_A\} = \mathcal{P}_A + \dot{Q}_A, \tag{18}$$

where $\mathrm{Tr}\{\hat{\mathcal{L}}_B(\hat{\rho}_s)\hat{\mathbf{H}}_A\}=0$. The first part is identified as the power and the second part as the heat current. This is motivated by the fact that the first part contains the interaction of atom A with the field mode, which is the work reservoir, and the second part expresses the interaction of atom A with the heat reservoir. This identification is the main result of [6]. Now, if the energy of a bipartite system is conserved (in the absence of dissipation), the energy of each subsystem is in general time dependent, but in this scheme there is no way to distinguish whether this is a result of an external force or because the subsystem is part of a larger bipartite system. We will return to these arguments in the next section.

The explicit form of the heat current and power for atom A is

$$\dot{Q}_A = 2\Delta E_A(\gamma_A^+ P_A^g - \gamma_A^- P_A^e),$$
$$\mathcal{P}_A = i\Delta E_A\lambda\,\mathrm{Tr}\{\hat{\rho}(\hat{\sigma}_A^-\hat{\sigma}_B^+\hat{a}^\dagger - \hat{\sigma}_A^+\hat{\sigma}_B^-\hat{a})\}, \tag{19}$$

where $P_e^A(P_g^A)$ is the occupation probability of the excited (ground) level for the atom A state. Similarly, for the atom B we have

$$\dot{Q}_B = 2\Delta E_B(\gamma_B^+ P_B^g - \gamma_B^- P_B^e),$$
$$\mathcal{P}_B = -i\Delta E_B\lambda\,\mathrm{Tr}\{\hat{\rho}(\hat{\sigma}_A^-\hat{\sigma}_B^+\hat{a}^\dagger - \hat{\sigma}_A^+\hat{\sigma}_B^-\hat{a})\}. \tag{20}$$

In general, $\dot{Q}_h = \dot{Q}_A + \dot{Q}_{Vh}$ and $\dot{Q}_c = \dot{Q}_B + \dot{Q}_{Vc}$, where $\dot{Q}_{Vh(c)} = \mathrm{Tr}\{\hat{\mathcal{L}}_{A(B)}(\hat{\rho}_s)\hat{H}_{int}\}$. However, our numerical results show that $\dot{Q}_{Vh(c)}=0$. For the cavity field we find

$$\dot{Q}_f = 0,$$
$$\mathcal{P}_f = -i\Omega\lambda\,\mathrm{Tr}\{\hat{\rho}_s(\hat{\sigma}_A^-\hat{\sigma}_B^+\hat{a}^\dagger - \hat{\sigma}_A^+\hat{\sigma}_B^-\hat{a})\}$$
$$= -i\Omega\lambda\{\langle\hat{\sigma}_A^-\hat{\sigma}_B^+\hat{a}^\dagger\rangle - \langle\hat{\sigma}_A^+\hat{\sigma}_B^-\hat{a}\rangle\}. \tag{21}$$

The heat engine condition is the same as the lasing condition Eq. (7). Under this condition and at "steady state" we find a heat current $\dot{Q}_A > 0$ flowing from the hot reservoir to the atom A and a heat current $\dot{Q}_B < 0$ from the atom B into the cold reservoir. These flows are accompanied by the previously mentioned amplification of the cavity field, i.e., power pumped into the field, so the system works as a heat engine. The second law of thermodynamics requires net nonnegative entropy production [49],

$$\frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} \geq 0. \tag{22}$$

The efficiency for the heat engine reads

$$\eta = \frac{P_f}{\dot{Q}_h} = 1 - \frac{\dot{Q}_c}{\dot{Q}_h} = 1 - \frac{\Delta E_B(\gamma_B^+ P_B^g - \gamma_B^- P_B^e)}{\Delta E_A(\gamma_A^+ P_A^g - \gamma_A^- P_A^e)}, \tag{23}$$

where we have used the conservation of energy (in steady state)

$$\mathcal{P}_f + \dot{Q}_h + \dot{Q}_c = 0. \tag{24}$$

Substituting Eq. (22) into Eq. (23) we get the Carnot inequality for the engine efficiency,

$$\eta \leq 1 - \frac{T_c}{T_h}. \tag{25}$$

Our numerical results confirm that the engine efficiency is always below the Carnot limit.

VII. WORK AND HEAT ACCORDING TO LEMBAS PRINCIPLE

The analysis in the previous section has completely been based on [6]. In so doing, we have ignored the conventional view of classical thermodynamics for a work source, which should have a constant entropy: This is not true in the present case [see Fig. 3(d)], and a similar result has been obtained in [6]. It is thus not sufficient to have the power defined via the Hamiltonian superoperator and heat flux defined via the dissipative superoperator to guarantee the corresponding thermodynamic concepts. Also claiming that there is no way of distinguishing whether the energy change of a subsystem within a bipartite system is a result of external forcing or because of the subsystem is part of a larger bipartite system, seems unjustified.

Along the line of [7] we now analyze the energy change of the field, taking the partial trace of Eq. (3) over the atomic system (AB) to get

$$\frac{\partial \hat{\rho}_f}{\partial t} = -i[\hat{H}_f + \hat{H}_f^{\text{eff}}, \hat{\rho}_f] + \hat{\mathcal{L}}_{inc}^{(f)}(\hat{\rho}), \tag{26}$$

where

$$\hat{H}_f^{\text{eff}} = \text{Tr}_{AB}\{\hat{H}_{int}(\hat{\rho}_{AB} \otimes \hat{1}_f)\},$$

$$\hat{\mathcal{L}}_{inc}^{(f)}(\hat{\rho}) = -i\ \text{Tr}_{AB}\{[\hat{H}_{int}, \hat{C}_{ABf}]\},$$

$$\hat{\rho}_s = \hat{\rho}_{AB} \otimes \hat{\rho}_f + \hat{C}_{ABf}. \tag{27}$$

Here $\hat{C}_{ABf}$ is the operator describing the correlations between matter and field. The reduced dynamics of the field specified by Eq. (26) is in a form that enables us to split its energy change into a part that does not alter the local von Neumann entropy and a part that does. It is easy to verify that the dynamics generated by $[\hat{H}_{int}, \hat{C}_{ABf}]$ cannot result in unitary dynamics, but will always change the local von Neumann entropy. We realize that for the first part of the density operator in the split form the factorization approximation [50] is exact. So we write

$$\text{Tr}_{AB}\{[\hat{H}_{int}, \hat{\rho}_{AB} \otimes \hat{\rho}_f]\} = [\hat{H}_f^{\text{eff}}, \hat{\rho}_f]. \tag{28}$$

Splitting the local energy change in this way is one part of the (LEMBAS) principle; the second feature is related to the fact that heat and work are defined by a process only, which in turn should be verified by observations. Observations are basis-dependent; here we choose the energy basis of the field (f) as the measurement basis, so that only the parts of the total effective Hamiltonian $\hat{H}_f^{\text{eff}}$ that commute with $\hat{H}_f$ would contribute to the described type of experiment [7] (testing the effective local energy changes). By expanding $\hat{H}_f^{\text{eff}}$ in the transition operator basis defined by the energy eigenstates of the field $|n\rangle$,

$$\hat{H}_f^{\text{eff}} = \sum_{jk} (\hat{H}_f^{\text{eff}})_{jk} |j\rangle\langle k|, \tag{29}$$

we can decompose $\hat{H}_f^{\text{eff}}$ into two parts,

$$\hat{H}_f^{\text{eff}} = \hat{H}_{f1}^{\text{eff}} + \hat{H}_{f2}^{\text{eff}}, \tag{30}$$

such that $[\hat{H}_{f1}^{\text{eff}}, \hat{H}_f]=0$ and $[\hat{H}_{f2}^{\text{eff}}, \hat{H}_f] \neq 0$, where

$$\hat{H}_{f1}^{\text{eff}} = \sum_{jj} (\hat{H}_f^{\text{eff}})_{jj} |j\rangle\langle j|,$$

$$\hat{H}_{f2}^{\text{eff}} = \hat{H}_f^{\text{eff}} - \hat{H}_{f1}^{\text{eff}}. \tag{31}$$

Now if a measurement of the field energy is performed in the energy eigenbasis of $\hat{H}_f$, the corresponding operator should be

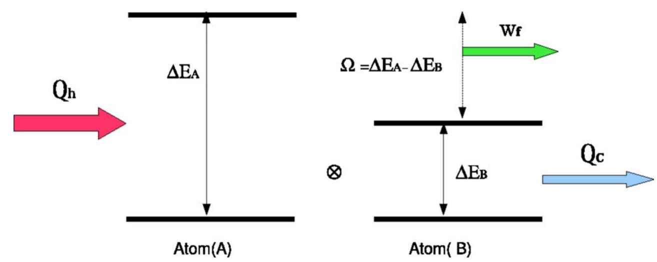

FIG. 6. (Color online) Schematic diagram of our model Fig. 1 working as a heat engine.

$$\hat{H}'_f = \hat{H}_f + \hat{H}_{f1}^{\text{eff}}. \tag{32}$$

Hence, the average change in the energy of the field is given by

$$\dot{E}'_f = \frac{d\langle \hat{H}'_f \rangle}{dt} = \frac{d}{dt}\text{Tr}_f\{\hat{\rho}_f \hat{H}'_f\} = \text{Tr}_f\{\dot{\hat{\rho}}_f \hat{H}'_f + \hat{\rho}_f \dot{\hat{H}}'_f\}. \tag{33}$$

Finally, we get the following expression for the power

$$P'_f = -i\Omega\lambda\{\langle \hat{\sigma}_A^- \hat{\sigma}_B^+ \rangle \langle \hat{a}^\dagger \rangle - \langle \hat{\sigma}_A^+ \hat{\sigma}_B^- \rangle \langle \hat{a} \rangle\}. \tag{34}$$

Using this (LEMBAS) formula, we find numerically that for various initial field states the power is identically zero except for starting from a coherent state. In this latter case a small amount of work (due to the initial coherency) shows up, which quickly disappears as the coherency decays. This behavior may be related to what has been found in [17,51]. Therefore, the (LEMBAS) formula tells us that the energy, which the field gains is heat not work.

The different results of the two approaches can be understood, as follows: in [6] we decide to consider the energy, which the field gains as work or useful energy, regardless of whether this energy gain is combined with entropy or not as long as the field is not directly linked to any heat reservoir. On the other hand, in [7], we get the effective dynamics of the field and stick with the classical thermodynamic point of view and identify work as that part of energy gained by the field, which does not correlate with a change in entropy. One should note the similarity between the two expressions for the power, Eqs. (21) and (34), which become identical, if $\hat{C}_{ABf}$=0, i.e., under the semiclassical or/and mean-field approximation. In former case we can consider each part of the engine "atoms and field" to behave as a classical driver for the other [22,52–54], e.g., by replacing the quantized field mode with a classical field.

The discrepancy between the two approaches, suggests that we need an additional test, whether or not that cavity field energy should count as useful work. As such a test we propose the heat pump operation.

VIII. HEAT PUMP OPERATION

A heat pump (see Fig. 6) is a device that transfers heat from a low-temperature reservoir to a high-temperature reservoir, i.e., against the temperature gradient, by applying external work. As the input of our machine we will use states similar to the outputs of our heat engine, namely, a single

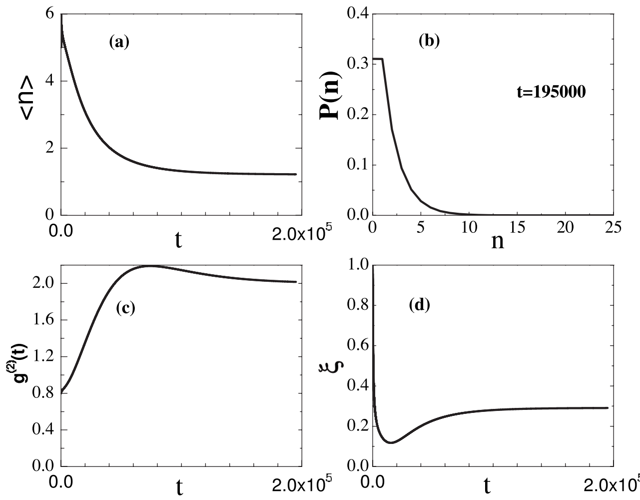

FIG. 7. Heat pump: the cavity field state as of Fig. 3 but for λ_E=0.001, λ=0.1, T_h=10, T_c=5, ΔE_A=8, and ΔE_B=1.

mode coherent state, thermal state, or a Fock state cavity field, respectively. In Fig. 7 we take the following parameters λ_E=0.001, λ=0.1, T_h=10, T_c=5, ΔE_A=8, ΔE_B=1, and initial Fock state $|5\rangle$ (also an initial coherent and thermal state with same average photon number shows almost the same behavior). Under the condition of positive effective temperature $P_{eg}<P_{ge}$ or, equivalently, $\frac{\Delta E_A}{\Delta E_B}>\frac{T_A}{T_B}>1$, and at a steady state we have $\dot{Q}_A\langle 0,\dot{Q}_B\rangle 0$, i.e., there is, indeed, a heat current from the cold reservoir to the hot reservoir, and the cavity field relaxes to an effective temperature given by

$$T_{eff}=\frac{-\hbar\Omega}{\ln\left(\frac{P_{eg}}{P_{ge}}\right)}>0. \tag{35}$$

The energy inside the cavity shows a continuous decrease toward a low energy equivalent to the effective temperature [Eq. (35)] [see Fig. 7(a)]. The second-order correlation function Fig. 7(c) indicates that asymptotically the field has thermal character. Also Fig. 7(b) shows that the photon statistics of the output is given by almost a Gaussian distribution, which is characteristic of a "thermal state." Therefore, the initial photon field operates as "useful work," irrespective of it being coherent or thermal. We remark that the entropy of the phase diffused Glauber state is smaller than that of the thermal state with the same energy.

It should be noted here that the link to the two baths establishes an appropriate population density of each atom and governs the occupation probabilities. The incoming radiation which is in resonance with the transition between the lasing levels change these occupation probabilities in such way that atom (A) looses energy to the hot bath and atom (B) accepts energy from the cold bath in order to maintain the population density required by each bath. This picture of a heat transfer against the temperature gradient relies on the selective coupling and not on the nature of the energy flow from the field (heat).

For further clarification we will simplify our model in two steps. In a first step we will reduce our model to a three-level

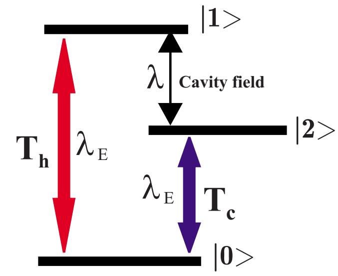

FIG. 8. (Color online) Three level system interacting with two heat reservoirs (hot and cold) and a quantized cavity mode.

quantum heat engine [3]. This model has been extensively studied [10,11,22,54]. Here, we will apply this model to compare two different heat pump scenarios based on frequency selective coupling: one making use of a cavity field as input (A) and a second one (B), for which the cavity field is replaced by a third heat bath.

A. Cavity field as input

We consider a three-level system interacting resonantly with one quantized cavity mode and two thermal reservoirs as schematically shown in Fig. 8. This model is qualitatively equivalent to the two-spin model as discussed before. In the interaction picture the system is described by the following Liouville–von Neumann equation,

$$\frac{\partial\hat{\rho}_s(t)}{\partial t}=-i[\hat{H}_{int},\hat{\rho}_s(t)]+\hat{\mathcal{L}}_h[\hat{\rho}_s(t)]+\hat{\mathcal{L}}_c[\hat{\rho}_s(t)]. \tag{36}$$

The Hamiltonian part of the Liouvillian is given by

$$\hat{H}_{int}=\lambda(\hat{\sigma}_{21}\otimes\hat{a}^{\dagger}+\hat{\sigma}_{21}^{\dagger}\otimes\hat{a}). \tag{37}$$

The above Hamiltonian is the interaction part of the standard Jaynes-Cummings model [55] of a single two-level atom interacting with a single-mode field, where the operators $\hat{a}^{\dagger}$ and $\hat{a}$ are the Bose creation and annihilation operators for the field mode, $\sigma_{21}=|2\rangle\langle 1|$ is the atomic transition operator, and λ is the matter-field coupling constant. The other two parts $\hat{\mathcal{L}}_h(\hat{\rho}_s)$ and $\hat{\mathcal{L}}_c(\hat{\rho}_s)$ are the dissipative hot and cold Lindblad superoperators, respectively,

$$\hat{\mathcal{L}}_h(\hat{\rho}_s)=\gamma_h^-([\hat{\sigma}_{01}\hat{\rho}_s,\hat{\sigma}_{01}^{\dagger}]+[\hat{\sigma}_{01},\hat{\rho}_s\hat{\sigma}_{01}^{\dagger}])+\gamma_h^+([\hat{\sigma}_{01}^{\dagger}\hat{\rho}_s,\hat{\sigma}_{01}]+[\hat{\sigma}_{01}^{\dagger},\hat{\rho}_s\hat{\sigma}_{01}]),$$

$$\hat{\mathcal{L}}_c(\hat{\rho}_s)=\gamma_c^-([\hat{\sigma}_{02}\hat{\rho}_s,\hat{\sigma}_{02}^{\dagger}]+[\hat{\sigma}_{02}\hat{\rho}_s\hat{\sigma}_{02}^{\dagger}])+\gamma_c^+([\hat{\sigma}_{02}^{\dagger}\hat{\rho}_s,\hat{\sigma}_{02}]+[\hat{\sigma}_{02}^{\dagger},\hat{\rho}_s\hat{\sigma}_{02}]). \tag{38}$$

The hot bath T_h couples to the transition between the excited level $|1\rangle$ and the ground level $|0\rangle$. The cold bath T_c couples to the transition between the intermediate level $|2\rangle$ and the ground level $|0\rangle$. Following Sec. II, we define the atom-bath coupling constants λ_E to be the same for the two baths, namely,

$$\lambda_E=\gamma_h^-+\gamma_h^+=\gamma_c^-+\gamma_c^+, \tag{39}$$

so we have

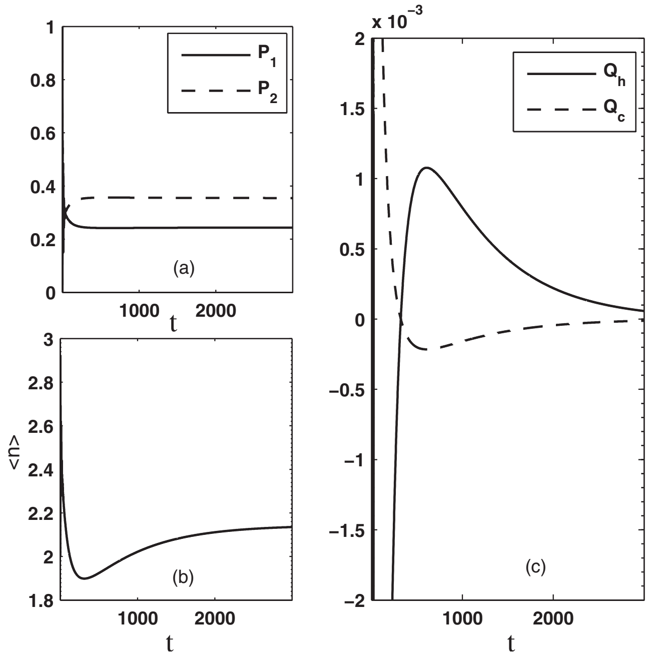

FIG. 9. (a) Level occupations, P_i, (b) average photon number, $\langle n\rangle$, and (c) heat currents, Q_i, for the three-level model. The initial field is in a Fock state with photon number $n=2$, the atom is initially in the excited level, $T_h=10$, $T_c=8$, $E_1=6$, $E_2=2$, and $E_0=1$. There is a short-time bouncing toward $\langle n\rangle=2.6$. Time is in inverse energy units $\times 10^3$.

$$\gamma^{+}_{h(c)} = \lambda_E \frac{1}{1+e^{\beta_{h(c)}\Delta E_{h(c)}}},$$

$$\gamma^{-}_{h(c)} = \lambda_E \frac{1}{1+e^{-\beta_{h(c)}\Delta E_{h(c)}}}, \tag{40}$$

where $\Delta E_{h(c)}=E_{1(2)}-E_0$. If we choose the parameters such that

$$1 < \frac{T_h}{T_c} \geq \frac{\Delta E_h}{\Delta E_c}, \tag{41}$$

the results of [22] are obtained, the field is amplified and we have a heat engine process. On the other hand, when

$$1 < \frac{T_h}{T_c} < \frac{\Delta E_h}{\Delta E_c}, \tag{42}$$

the field is damped and we have the heat pump process, which we will discuss here. Along the same line of Sec. V we get the following expression of the heat currents:

$$\dot{Q}_h = 2\Delta E_h(\gamma^{+}_h P_0 - \gamma^{-}_h P_1),$$

$$\dot{Q}_c = 2\Delta E_c(\gamma^{+}_c P_0 - \gamma^{-}_c P_2). \tag{43}$$

Figures 9–11 show the corresponding heat currents Q_h, Q_c between the two baths and the three-level system. A positive heat current indicates that the current is from the bath to the three-level system. The three-level occupation probabilities P_1 and P_2 for the excited and intermediate levels, respectively, are also shown. We find that whenever the initial photon number (Fock state) or the average photon number (Glauber state, thermal state, phase-diffused Glauber state, or any single mode cavity field) is larger than $\langle n_{st}\rangle$, there is a heat current from the cold bath to the hot bath (heat pump process), and if it is less than $\langle n_{st}\rangle$, the direction of currents is reversed. Here, $\langle n_{st}\rangle$ is the stationary average photon number of the field given by $\langle n_{st}\rangle=(\frac{P_2}{P_1}-1)^{-1}$. This is the average photon number corresponding to a thermal state with temperature

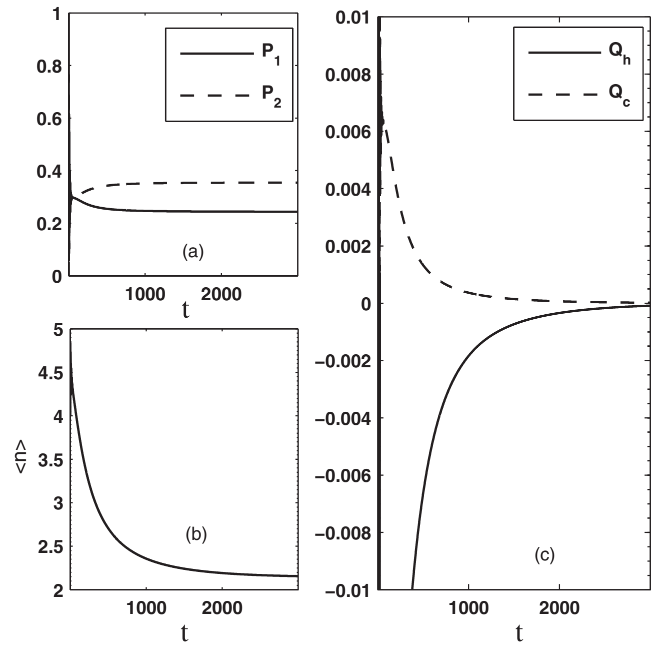

FIG. 10. Same as Fig. 9, but with the field initially in a Fock state with $n=4$. There is a short-time bouncing toward $\langle n\rangle=4.5$.

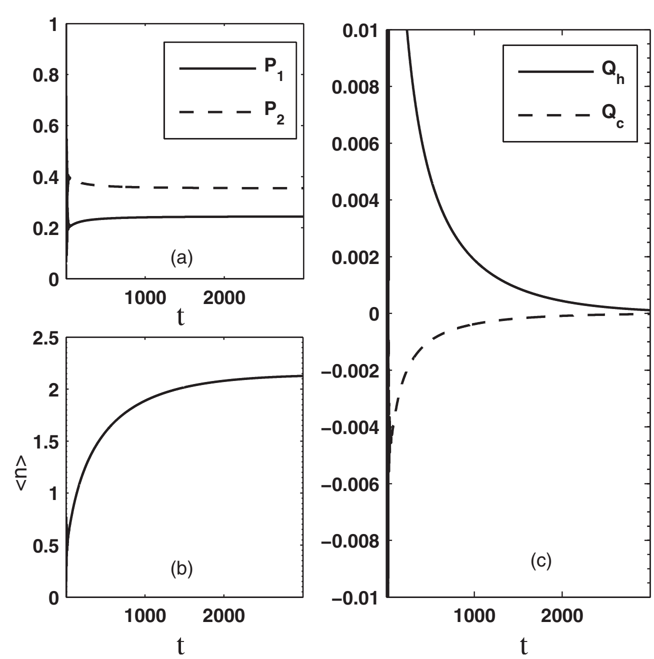

FIG. 11. Same as Fig. 9, but with the field initially in a Glauber state with average photon number $\langle n\rangle=0.001$.

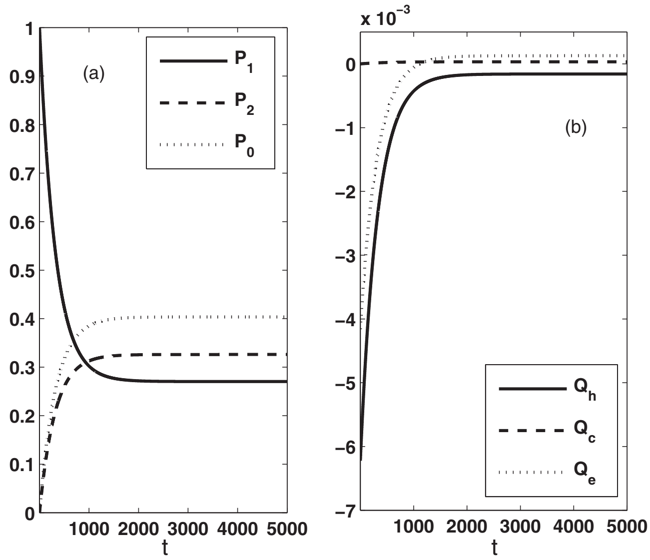

FIG. 12. (a) Level occupations, P_i, and (b) heat currents, Q_i, for the three-level–three-bath heat engine model. $T_h=10$, $T_c=8$, $E_1=6$, $E_2=2$, $E_0=1$, and $T_e=50>T_{eff}=10.667$. Time is in inverse energy units $\times 10^3$.

$$T_{eff} = \frac{-\Omega_f}{\ln\left(\frac{P_1}{P_2}\right)}. \tag{44}$$

For the parameters used in the previous figures $\langle n_{st}\rangle=2.197$. This indicates that this stationary average photon number $\langle n_{st}\rangle$ is the threshold for the phase-diffused Glauber state and for any other single mode input field. The direction of currents are reversed not only when we let the average photon number within a Glauber state go to zero, Fig. 11, but also for any single mode cavity field state with average photon number less than $\langle n_{st}\rangle$. Glauber states with $\langle n_{st}\rangle=2,4$ show a relaxation behavior virtually identical with that of the corresponding Fock states.

B. Auxiliary bath as input

Now we take the same three-levels system, but replace the photon field connecting the excited state $|1\rangle$ with the intermediate $|2\rangle$ level by a third auxiliary bath T_e. The system is governed by the following master equation in the interaction picture,

$$\frac{\partial \hat{\rho}_s(t)}{\partial t} = \hat{\mathcal{L}}_e[\hat{\rho}_s(t)] + \hat{\mathcal{L}}_h[\hat{\rho}_s(t)] + \hat{\mathcal{L}}_c[\hat{\rho}_s(t)]. \tag{45}$$

$\hat{\mathcal{L}}_e(\hat{\rho}_s)$ is the third bath Lindblad superoperator,

$$\hat{\mathcal{L}}_e(\hat{\rho}_s) = \gamma_e^-([\hat{\sigma}_{21}\hat{\rho}_s,\hat{\sigma}_{21}^\dagger] + [\hat{\sigma}_{21},\hat{\rho}_s\hat{\sigma}_{21}^\dagger]) + \gamma_e^+([\hat{\sigma}_{21}^\dagger\hat{\rho}_s,\hat{\sigma}_{21}] + [\hat{\sigma}_{21}^\dagger,\hat{\rho}_s\hat{\sigma}_{21}]). \tag{46}$$

Figures 12 and 13 show the resulting heat currents Q_h, Q_c, and Q_e between the three baths and the three-level system, the occupation probabilities P_1, P_2, and P_0 for the excited, intermediate, and ground states, respectively. We distinguish the following two cases.

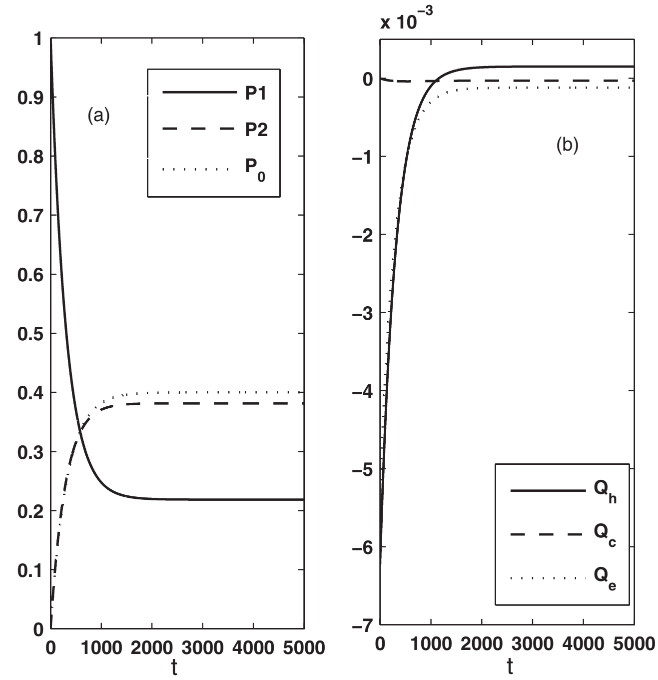

FIG. 13. As Fig. 12 but with $T_e=6<T_{eff}=10.667$.

Case (1)—whenever $T_e>T_{eff}$, there is a heat current from the bath T_e'', $Q_e>0''$ to the three-level system, from the bath T_c'', $Q_c>0''$ to the three-level system, and from the three-level to the bath T_h'', $Q_h<0''$ (see Fig. 12).

Case (2)—whenever $T_e<T_{eff}$, there is a heat current from the bath T_h'', $Q_h<0''$ to the three-level system, from the three-level to the bath T_c'', $Q_c>0''$, and to the bath T_e'', $Q_e>0''$ (see Fig. 13). The third bath can thus drive the heat pump operation, quite similar to the incoherent Glauber state studied previously.

What can be concluded from these results? Obviously, for all those models considered, the directionality of energy flows is intimately related to frequency-selective couplings rather than to the conventional notions of heat versus work. For an additional characterization of the underlying process the LEMBAS-definitions appear to be preferable, as they refer to the local entropy balance in a most transparent way.

IX. SUMMARY AND CONCLUSIONS

We have studied an open two-atom scenario internally coupled by a resonant cavity photon field and externally to two noninteracting heat baths at different temperatures. Our two-level two atom model is functionally equivalent to a single three-level center; however, the composite nature of the former allows a spatial separation and thus should simplify the frequency selective coupling to the two different heat baths, T_h and T_c (at opposite ends of the cavity).

The system can operate as a laser or as a heat pump. We have carefully characterized the resulting cavity field under lasing condition. It is a single frequency, but basically incoherent (phase-diffused Glauber) state. As previously argued by Molmer, such an incoherent field should be indistinguishable from a coherent state with respect to standard quantum optical experiments. Nevertheless, a more detailed analysis in terms of the "usefulness" of this photon field in terms of thermodynamic considerations seems highly desirable.

The splitting of energy change into work and heat applies to the respective process involved, not to any momentary state as such. Here, we have compared two recently suggested definitions. Most remarkably we found that these two approaches give completely different results: According to LEMBAS, the energy transfer between matter (atom pair A-B) and field should be heat only, not work. While in the alternative method due to Boukobza and Tannor that energy transfer would have to count as work—despite the accompanying change of entropy.

This ambiguity has finally led us to supplement our investigations by a heat pump scenario, in which the above laser output states would have to show their respective work value as input. We verified that also this heat pump functionality does not depend on coherence; i.e., the Glauber and the phase-diffused Glauber states, e.g., work equally well. In our attempt to locate the essential reasons for that unexpected behavior we further simplified our original model in two steps: based on a reduced three-level model, we found that for such a heat pump operation the single frequency photon field could even be substituted by a simple heat bath of a temperature T_e above a certain threshold temperature T_{eff} combined with frequency-selective coupling. This coupling is nonergodic in the sense that relaxation depends on the initial preparation. In the case of our full original model the frequency selective coupling operates like a kind of switch between the cavity field and the two external heat baths. It is this structure—alien to any classical machinery—which guarantees the desired functionality. What remains from thermodynamics are the underlying relaxation phenomena. Beyond that notion thermodynamic concepts can become ambiguous and should be used with care.

ACKNOWLEDGMENTS

We thank H. Schröder, M. J. Henrich, J. Teifel, Th. Jahnke, G. Waldherr, and J. Birjukov for fruitful discussions. Also one of us (M.Y.) wished to thank the members of the Institute of Theoretical Physics for their most generous hospitality during his stay at the University of Stuttgart.

APPENDIX A: DENSITY MATRIX ELEMENTS EQUATIONS IN THE ATOM-FOCK BASIS

In this appendix we convert operator (3) into an infinite set of coupled ordinary differential equations by taking matrix elements with respect to the appropriate basis. Let $|n\rangle$, $n=0,1,2,\ldots$, be a Fock photon number basis and $|\alpha,\beta\rangle$, $\alpha(\beta)=e,g$, be the states of the atoms A and B being in the state α,β. Using this basis and the following notation (we will drop the subscript s when referring to the elements of system density operator):

$$\dot{\rho}^{n;m}_{\alpha\beta;\tilde{\alpha}\tilde{\beta}}=\langle n,\alpha,\beta|\dot{\hat{\rho}}_s(t)|m,\tilde{\alpha},\tilde{\beta}\rangle. \tag{A1}$$

Equation (3) produces the following set of density matrix elements equations

$$\dot{\rho}^{n;m}_{ee;ee}=2\gamma_A^{(+)}\rho^{n;m}_{ge;ge}+2\gamma_B^{(+)}\rho^{n;m}_{eg;eg}-2(\gamma_A^{(-)}+\gamma_B^{(-)})\rho^{n;m}_{ee;ee}, \tag{A2}$$

$$\dot{\rho}^{n,m}_{eg;eg}=-i\lambda\sqrt{n+1}\rho^{n+1;m}_{ge;eg}+i\lambda\sqrt{m+1}\rho^{n;m+1}_{eg;ge}-2\gamma_A^{(-)}\rho^{n;m}_{eg;eg}+2\gamma_A^{(+)}\rho^{n;m}_{gg;gg}+2\gamma_B^{(-)}\rho^{n;m}_{ee;ee}-2\gamma_B^{(+)}\rho^{n;m}_{eg;eg}, \tag{A3}$$

$$\dot{\rho}^{n+1;m+1}_{ge;ge}=-i\lambda\sqrt{n+1}\rho^{n;m+1}_{eg;ge}+i\lambda\sqrt{m+1}\rho^{n+1;m}_{ge;eg}+2\gamma_A^{(-)}\rho^{n+1,m+1}_{ee;ee}-2\gamma_A^{(+)}\rho^{n+1;m+1}_{ge;ge}-2\gamma_B^{(-)}\rho^{n+1;m+1}_{ge;ge}+2\gamma_B^{(+)}\rho^{n+1;m+1}_{gg;gg}, \tag{A4}$$

$$\dot{\rho}^{n;m}_{gg;gg}=2\gamma_A^{(-)}\rho^{n;m}_{eg;eg}+2\gamma_B^{(-)}\rho^{n;m}_{ge;ge}-2(\gamma_A^{(+)}+\gamma_B^{(+)})\rho^{n;m}_{gg;gg}, \tag{A5}$$

$$\dot{\rho}^{n;m}_{ee;eg}=i\lambda\sqrt{m+1}\rho^{n;m+1}_{ee;ge}-2\gamma_A^{(-)}\rho^{n;m}_{ee;eg}+2\gamma_A^{(+)}\rho^{n;m}_{ge;gg}-\gamma_B^{(-)}\rho^{n;m}_{ee;eg}-\gamma_B^{(+)}\rho^{n;m}_{ee;eg}, \tag{A6}$$

$$\dot{\rho}^{n;m+1}_{ee;ge}=i\lambda\sqrt{m+1}\rho^{n;m}_{ee;eg}-\gamma_A^{(-)}\rho^{n;m+1}_{ee;ge}-\gamma_A^{(+)}\rho^{n;m+1}_{eg;ge}-2\gamma_B^{(-)}\rho^{n;m+1}_{ee;ge}+2\gamma_B^{(+)}\rho^{n;m+1}_{eg;gg}, \tag{A7}$$

$$\dot{\rho}^{n;m}_{ee;gg}=-(\gamma_A^{(-)}+\gamma_B^{(-)}+\gamma_A^{(+)}+\gamma_B^{(+)})\rho^{n;m}_{ee;gg}, \tag{A8}$$

$$\dot{\rho}^{n;m+1}_{eg;ge}=-i\lambda\sqrt{n+1}\rho^{n+1;m+1}_{ge;ge}+i\lambda\sqrt{m+1}\rho^{n;m}_{eg;eg}-\gamma_A^{(-)}\rho^{n;m+1}_{eg;ge}-\gamma_A^{(+)}\rho^{n;m+1}_{eg;ge}-\gamma_B^{(-)}\rho^{n;m+1}_{eg;ge}-\gamma_B^{(+)}\rho^{n;m+1}_{eg;ge}, \tag{A9}$$

$$\dot{\rho}^{n;m}_{eg;gg}=-i\lambda\sqrt{n+1}\rho^{n+1;m}_{ge;gg}-\gamma_A^{(-)}\rho^{n;m}_{eg;gg}-\gamma_A^{(+)}\rho^{n;m}_{eg;gg}+2\gamma_B^{(-)}\rho^{n;m}_{ee;ge}-2\gamma_B^{(+)}\rho^{n;m}_{eg;gg}, \tag{A10}$$

$$\dot{\rho}^{n+1;m}_{ge;gg}=-i\lambda\sqrt{n+1}\rho^{n;m}_{eg;gg}+2\gamma_A^{(-)}\rho^{n+1;m}_{ee;eg}-(2\gamma_A^{(+)}+\gamma_B^{(-)}+\gamma_B^{(+)})\rho^{n+1;m}_{ge;gg}. \tag{A11}$$

APPENDIX B: LASER EQUATION

The equation of motion for the laser field density matrix due to the interaction with active lasing medium (the two atoms) and the pumping mechanism is given by taking the trace over the atomic basis $\{|e,e\rangle,|e,g\rangle,|g,e\rangle,|g,g\rangle\}$ of the atom-field density matrix (3),

$$\begin{aligned}\dot{\rho}^{n,m}_f&=\langle n|\dot{\rho}_f|m\rangle=\langle n|\mathrm{Tr}_{atoms}[\dot{\hat{\rho}}_s(t)]|m\rangle\\&=-i(\mathrm{Tr}_{AB}\{[\hat{H}_S,\hat{\rho}_s(t)]\})_{n,m}\\&\quad+(\mathrm{Tr}_{AB}\{\hat{\mathcal{L}}_D[\hat{\rho}_s(t)]\})_{n,m}\\&=(\dot{\rho}^{n,m}_f)_{gain}+(\dot{\rho}^{n,m}_f)_{loss}.\end{aligned} \tag{B1}$$

The second part is identically zero, since this term is only directly affected by the atomic system. Therefore, we have

$$\dot{\rho}^{n,m}_f=-i\lambda\{\sqrt{n}\rho^{n-1;m}_{eg;ge}+\sqrt{n+1}\rho^{n+1;m}_{ge;eg}-\sqrt{m+1}\rho^{n;m+1}_{eg;ge}-\sqrt{m}\rho^{n;m-1}_{ge;eg}\}. \tag{B2}$$

The equations of motion for the relevant density matrix elements are the same set of equations as listed in Appendix A. In addition we have the following trivial equation:

$$\rho_f^{n;m} = \langle n|\mathrm{Tr}_{AB}\{\hat{\rho}_s(t)\}|m\rangle = \rho_{ee;ee}^{n;m} + \rho_{eg;eg}^{n;m} + \rho_{ge;ge}^{n;m} + \rho_{gg;gg}^{n;m}. \tag{B3}$$

This is all we need to derive the laser equation. The steady state solutions to Eqs. (A2) and (A5) are

$$\rho_{ee;ee}^{n;m} = \frac{\gamma_A^{(+)}}{\gamma_{AB}^{(-)}}\rho_{ge;ge}^{n;m} + \frac{\gamma_B^{(+)}}{\gamma_{AB}^{(-)}}\rho_{eg;eg}^{n;m}, \tag{B4}$$

$$\rho_{gg;gg}^{n;m} = \frac{\gamma_A^{(-)}}{\gamma_{AB}^{(+)}}\rho_{eg;eg}^{n;m} + \frac{\gamma_B^{(-)}}{\gamma_{AB}^{(+)}}\rho_{ge;ge}^{n;m}, \tag{B5}$$

where

$$\gamma_{AB}^{(+)} = (\gamma_A^{(+)} + \gamma_B^{(+)}),$$

$$\gamma_{AB}^{(-)} = (\gamma_A^{(-)} + \gamma_B^{(-)}). \tag{B6}$$

Substituting from Eqs. (B5) and (B4) into Eq. (B3) we get

$$\rho_f^{n;m} = A\rho_{eg;eg}^{n;m} + B\rho_{ge;ge}^{n;m}, \tag{B7}$$

where

$$A = \frac{\gamma_A^{(-)}\gamma_{AB}^{(-)} + \gamma_B^{(+)}\gamma_{AB}^{(+)}}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}} + 1,$$

$$B = \frac{\gamma_B^{(-)}\gamma_{AB}^{(-)} + \gamma_A^{(+)}\gamma_{AB}^{(+)}}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}} + 1. \tag{B8}$$

Substituting Eqs. (B5) and (B4) into Eq. (A3) we get

$$\dot{\rho}_{eg;eg}^{n,m} = -i\lambda\sqrt{n+1}\rho_{ge;eg}^{n+1;m} + i\lambda\sqrt{m+1}\rho_{eg;ge}^{n;m+1} + M_1\rho_{eg;eg}^{n;m} + N_1\rho_{ge;ge}^{n;m}, \tag{B9}$$

where

$$M_1 = \frac{2\gamma_B^{(-)}\gamma_B^{(+)}\gamma_{AB}^{(+)} + 2\gamma_A^{(+)}\gamma_A^{(-)}\gamma_{AB}^{(-)} - \chi_1}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}},$$

$$N_1 = \frac{2\gamma_B^{(-)}\gamma_A^{(+)}\gamma_{AB}^{(+)} + 2\gamma_A^{(+)}\gamma_B^{(-)}\gamma_{AB}^{(-)}}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}},$$

$$\chi_1 = 2\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}(\gamma_A^{(-)} + \gamma_B^{(+)}). \tag{B10}$$

Using Eq. (B7) to eliminate $\rho_{ge;ge}^{n;m}$ in Eq. (B9) we get

$$\dot{\rho}_{eg;eg}^{n,m} = -i\lambda\sqrt{n+1}\rho_{ge;eg}^{n+1;m} + i\lambda\sqrt{m+1}\rho_{eg;ge}^{n;m+1} + \left(\frac{M_1B - N_1A}{B}\right)\rho_{eg;eg}^{n;m} + \frac{N_1}{B}\rho_{n;m}. \tag{B11}$$

The solution of Eq. (A9) at steady state is given by

$$\rho_{eg;ge}^{n;m+1} = \frac{i\lambda}{2\lambda_E}(\sqrt{m+1}\rho_{eg;eg}^{n;m} - \sqrt{n+1}\rho_{ge;ge}^{n+1;m+1}). \tag{B12}$$

Substituting Eq. (B12) into Eq. (B11), solving for steady state by setting ($\dot{\rho}_{eg;eg}^{n,m}=0$) and considering only the diagonal elements ($n=m$), we have the following equation:

$$\rho_{eg;eg}^{n,n} = \frac{N_1}{N_1A - M_1B}\rho_f^{n;n} + \frac{B\bar{\lambda}(n+1)}{(N_1A - M_1B)}(\rho_{eg;eg}^{n,n} - \rho_{ge;ge}^{n+1,n+1}). \tag{B13}$$

Following a similar procedure and using Eqs. (B4), (B5), (B7), and (B12) as well as Eq. (A4) we find

$$\rho_{ge;ge}^{n+1,n+1} = \frac{N_2}{N_2B - M_2A}\rho_f^{n+1;n+1} + \frac{A\bar{\lambda}(n+1)}{(N_2B - M_2A)}(\rho_{eg;eg}^{n,n} - \rho_{ge;ge}^{n+1,n+1}), \tag{B14}$$

where

$$M_2 = \frac{2\gamma_A^{(-)}\gamma_A^{(+)}\gamma_{AB}^{(+)} + 2\gamma_B^{(+)}\gamma_B^{(-)}\gamma_{AB}^{(-)} - \chi_2}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}},$$

$$N_2 = \frac{2\gamma_A^{(-)}\gamma_B^{(+)}\gamma_{AB}^{(+)} + 2\gamma_A^{(-)}\gamma_B^{(+)}\gamma_{AB}^{(-)}}{\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}},$$

$$\chi_2 = 2\gamma_{AB}^{(-)}\gamma_{AB}^{(+)}(\gamma_A^{(+)} + \gamma_B^{(-)}),$$

$$\bar{\lambda} = \frac{\lambda^2}{\lambda_E}. \tag{B15}$$

From Eqs. (B13) and (B14) we get

$$\rho_{eg;eg}^{n,n} - \rho_{ge;ge}^{n+1,n+1} = \Pi_1\rho_f^{n,n} - \Pi_2\rho_f^{n+1,n+1}, \tag{B16}$$

where

$$\Pi_1 = \frac{N_1(N_2B - M_2A)}{(N_1A - M_1B)(N_2B - M_2A) + \chi_3},$$

$$\Pi_2 = \frac{N_2(N_1A - M_1B)}{(N_1A - M_1B)(N_2B - M_2A) + \chi_3},$$

$$\chi_3 = \bar{\lambda}(n+1)\{B(N_2B - M_2A) + A(N_1A - M_1B)\}.$$

Substituting Eq. (B16) into Eq. (B12)

$$\rho_{ge;ge}^{n;n+1} = \frac{i\lambda\sqrt{n+1}}{2\lambda_E}(\Pi_1\rho_{n;n} - \Pi_2\rho_{n+1;n+1}). \tag{B17}$$

Substituting this equation into Eq. (B2) and considering the diagonal elements ($n=m$) we get the required laser equation,

$$\dot{P}(n) = \dot{\rho}_f^{n;n} = \bar{\lambda}\Pi_1 nP(n-1) - \bar{\lambda}\Pi_2 nP(n) - \bar{\lambda}\Pi_1(n+1)P(n) + \bar{\lambda}\Pi_2(n+1)P(n+1).$$

Here $P(n)$ is the probability of having n photons in the field at a given time.

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